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COMMENT

Comment on 'Solutions of the Yang–Baxter equation for isotropic quantum spin chains'

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Abstract. We comment on a recent paper by Kennedy (*J. Phys. A: Math. Gen.* **25** (1992) 2809) in which a systematic search for integrable spin-S $su(2)$-invariant quantum chains for $S \leqslant 6$ revealed four spin-S families of integrable chains along with an additional integrable chain at $S = 3$. We identify these $su(2)$-invariant chains with known $\mathcal{G}$-invariant R-matrices, where $\mathcal{G}$ is a simple Lie algebra, and give arguments that Kennedy's results may well constitute the complete classification of integrable spin-S $su(2)$-invariant chains.

In a recent paper [1] Kennedy initiated a systematic search for spin-S $su(2)$-invariant quantum chains satisfying Reshetikhin's condition [1, 2], which is necessary for integrability. Four known models for generic S were presented, together with the corresponding R-matrices satisfying the Yang–Baxter equation. These four families of quantum chains were located in the numerical analysis performed for spins $S \leqslant 6$, together with an extra solution at $S = 3$ for which the related R-matrix was also given.

In this comment we discuss the relationship of these $su(2)$-invariant spin chains to known $\mathcal{G}$-invariant R-matrices, where $\mathcal{G}$ is a simple Lie algebra. This aspect was briefly considered in [1]. Indeed, we will show that the identification made there of one 'family' of quantum chains with the $so(n)$-invariant R-matrices of Zamolodchikov and Zamolodchikov [3] is only half correct; the correct identification being that of the spin-S member for S integer with the $so(2S+1)$ R-matrix and the spin-S member for S half-odd-integer with the $sp(2S+1)$ R-matrix.

We will adopt the root labelling convention of Dynkin [4], and for $\mathcal{G}$ a simple Lie algebra of rank n denote the fundamental weights by $\Lambda_1, \ldots, \Lambda_n$. Let π_Λ be an irreducible representation (irrep) of $\mathcal{G}$ with highest weight Λ on the vector space V_Λ. The R-matrix $\check{R}^{\Lambda,\Lambda}(u) \in \mathrm{End}\,(V_\Lambda \otimes V_\Lambda)$ is said to be $\mathcal{G}$-invariant if

$$[\ \check{R}^{\Lambda,\Lambda}(u),\ \pi_\Lambda(\mathcal{G}) \otimes 1 + 1 \otimes \pi_\Lambda(\mathcal{G})\ ] = 0.$$

For a given pair $(\mathcal{G}, \Lambda)$ the imposition of such a condition sometimes (but not always [5, 6]) allows the Yang–Baxter equation for $\check{R}^{\Lambda,\Lambda}(u)$ to be solved. In particular, for any $\mathcal{G}$ (except E_8) and Λ corresponding to the lowest dimensional irreps the solutions are known explicitly. In this language, $su(2)$-invariance in the sense of [1] corresponds to requiring $\mathcal{G} = su(2)$ and $\Lambda = 2S\Lambda_1$. Note that we have left out the label for $\mathcal{G}$ in $\check{R}^{\Lambda,\Lambda}(u)$, which is a usual practice. More seriously, the results of [1] show that, in general, an extra label is required to take into account the possibility of more than one solution for given $(\mathcal{G}, \Lambda)$. Accordingly, we will call the four families of $su(2)$-invariant solutions in [1] $\check{R}_i^{2S\Lambda_1,2S\Lambda_1}$ where $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$.

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An R-matrix $\check{R}^{\Lambda,\Lambda}(u)$ corresponding to $(\mathcal{G},\Lambda)$ has the spectral decomposition $\check{R}^{\Lambda,\Lambda}(u) = \sum_{\lambda\in\mathcal{D}} c_\lambda(u) P_\lambda$, where P_λ is a projector onto the irreducible subspace V_λ occurring in the Clebsch–Gordan decomposition $V_\Lambda \otimes V_\Lambda = \oplus_{\lambda\in\mathcal{D}} V_\lambda$. Such an R-matrix turns out also to be $su(2)$-invariant in the sense of [1] if the following condition is satisfied:

(a) The space V_Λ can be identified with a space $V_{2S\Lambda_1}$ on which $su(2)$ is represented *irreducibly*.

We are unable to give a complete classification of all pairs $(\mathcal{G},\Lambda)$ such that this condition holds. However, by an examination of the tables of branching rules [4] for simple Lie algebras of rank $\leqslant 8$ and representations of dimension < 5000, we have found the following solutions:

(i) $(A_n = su(n+1), \Lambda_1)$ for $n \geqslant 1$,
(ii) $(B_n = so(2n+1), \Lambda_1)$ for $n \geqslant 3$,
(iii) $(C_n = sp(2n), \Lambda_1)$ for $n \geqslant 2$, and
(iv) $(G_2, \Lambda_2, \Lambda_2)$.

Before proceeding further, we rewrite the $su(2)$-invariant R-matrices of [1] in spectral form, using the relation $\mathcal{P} = (-1)^{2S}\sum_{i=0}^{2S}(-1)^i P^{(i)}$ between the permutation operator $\mathcal{P}$ (E in the notation of [1]) and projection operators $P^{(j)} \equiv P_{2j\Lambda_1}$ onto $su(2)$-irreps in $V_{2S\Lambda_1} \otimes V_{2S\Lambda_1}$. The results are :

$$\check{R}_{\rm I}^{2S\Lambda_1,2S\Lambda_1}(u) = (1-u)\sum_{i\ \rm even} P^{(i)} + (1+u)\sum_{i\ \rm odd} P^{(i)} \tag{1}$$

$$\begin{aligned}\check{R}_{\rm IIa}^{2S\Lambda_1,2S\Lambda_1}(u) &= (1-u)\left(1-(S-\tfrac{1}{2})u\right)P^{(0)} + (1+u)\left(1-(S-\tfrac{1}{2})u\right)\sum_{i\ \rm odd} P^{(i)} \\ &\quad + (1+u)\left(1+(S-\tfrac{1}{2})u\right)\sum_{i\ \rm even \neq 0} P^{(i)} \qquad (S \text{ integer})\end{aligned} \tag{2}$$

$$\begin{aligned}\check{R}_{\rm IIb}^{2S\Lambda_1,2S\Lambda_1}(u) &= (1-u)\left(1+(S+\tfrac{3}{2})u\right)P^{(0)} + (1+u)\left(1+(S+\tfrac{3}{2})u\right)\sum_{i\ \rm odd} P^{(i)} \\ &\quad + (1+u)\left(1-(S+\tfrac{3}{2})u\right)\sum_{i\ \rm even \neq 0} P^{(i)} \qquad (S \text{ half odd integer})\end{aligned} \tag{3}$$

$$\check{R}_{\rm III}^{2S\Lambda_1,2S\Lambda_1}(u) = \sum_{k=0}^{2S}\left(\prod_{j=1}^{k}(j-u)\prod_{j=k+1}^{2S}(j+u)\right)P^{(k)} \tag{4}$$

$$\check{R}_{\rm IV}^{2S\Lambda_1,2S\Lambda_1}(u) = 1 + \frac{a-ae^u}{e^u-a^2}(2S+1)P^{(0)} \qquad a+\tfrac{1}{a} = 2S+1 \qquad (S \geqslant 1). \tag{5}$$

Here, it is understood that $\sum_{i\ \rm even}$ is short for $\sum_{i=0\ (i\ \rm even)}^{2S}$ etc. The R-matrices labelled I, II, III and IV correspond, respectively, to the solutions (4), (6), (9) and (10) in [1]. We have divided the type II solutions into IIa and IIb for reasons which will soon be clear. The extra solution for $S = 3$ ((13) in [1]) can be written as

$$\begin{aligned}\check{R}_{\rm V}^{6\Lambda_1,6\Lambda_1}(u) &= (1+6u)(1+u)(1-\tfrac{3}{2}u)P^{(0)} + (1-6u)(1-u)(1+\tfrac{3}{2}u)P^{(3)} \\ &\quad + (1-6u)(1-u)(1-\tfrac{3}{2}u)\left(P^{(2)}+P^{(4)}+P^{(6)}\right) \\ &\quad + (1+6u)(1-u)(1-\tfrac{3}{2}u)\left(P^{(1)}+P^{(5)}\right).\end{aligned} \tag{6}$$

The solutions of the Yang–Baxter equation corresponding to the pairs $(\mathcal{G},\Lambda)$ written out above which satisfy condition (a) can be found in [7, 8] as scattering matrices in spectral form, which can be converted to regular R-matrices (satisfying $\check{R}^{\Lambda,\Lambda}(0) = 1$).† For the $su(k)$ and $so(2k+1)$ series, we write down the results only for generic k; for the special low-dimensional cases $su(2)$, $so(3)$, and $so(5)$, the corresponding R-matrices have similar forms to the generic rank cases but the labelling of the invariant subspaces is slightly different. We now examine each case (i)-(iv) more closely. In the tables of [4] only branching rules corresponding to maximal embeddings [9] are given; therefore in arriving at the list (i)-(iv) we have examined all chains of maximal embeddings $\mathcal{G} \supset \cdots \supset A_1$. Consideration of such chains is also needed in general to relate the corresponding R-matrices to those in (1)–(5).

In case (i) the R-matrices are given by

$$\check{R}^{\Lambda_1,\Lambda_1}(u) = (1-u)P_{\Lambda_2} + (1+u)P_{2\Lambda_1}. \tag{7}$$

The required embeddings of A_1 in A_n are $A_{2k} \supset B_k \supset A_1$ and $A_{2k-1} \supset C_k \supset A_1$, except possibly when an exceptional algebra 'gets in the way', e.g. $A_6 \supset B_3 \supset G_2 \supset A_1$. In this latter case the relevant branching rules are $\Lambda_1 \downarrow (3)$, $\Lambda_2 \downarrow (1) + (3) + (5)$, and $2\Lambda_1 \downarrow (0) + (2) + (4) + (6)$, where $su(2)$-irreps are labelled by their spin. Hence, the corresponding R-matrix can be identified with the $S = 3$ case of (1). Likewise, examination of the other cases will show that the R-matrices (7) are identified with those of (1). Of course, one can arrive at the same conclusion by noting that both series of R-matrices can be expressed in terms of the permutation operator $\mathcal{P}$. The approach taken here shows its utility when we consider the other Lie algebras.

In case (ii) the R-matrices are

$$\begin{aligned}\check{R}^{\Lambda_1,\Lambda_1}(u) = {} & (1+u)\left(1+(n-\tfrac{1}{2})u\right)P_{2\Lambda_1} + (1+u)\left(1-(n-\tfrac{1}{2})u\right)P_{\Lambda_2} \\ & +(1-u)\left(1-(n-\tfrac{1}{2})u\right)P_0.\end{aligned} \tag{8}$$

The required embedding is generically $B_n \supset A_1$. For example, we have $B_4 \supset A_1$, with the relevant branching rules $\Lambda_1 \downarrow (4)$, $\Lambda_2 \downarrow (1)+(3)+(5)+(7)$, and $2\Lambda_1 \downarrow (2)+(4)+(6)+(8)$. The corresponding R-matrix is identified with the $S = 4$ member of (2). Likewise, all the other $so(2n+1)$-invariant R-matrices (8) can be identified with members of the series (2).

In case (iii) the R-matrices are

$$\begin{aligned}\check{R}^{\Lambda_1,\Lambda_1}(u) = {} & (1+u)\,(1+(n+1)u)\,P_{2\Lambda_1} + (1+u)\,(1-(n+1)u)\,P_{\Lambda_2} \\ & +(1-u)\,(1+(n+1)u)\,P_0.\end{aligned} \tag{9}$$

The required embedding is generically $C_n \supset A_1$. For example, we have $C_3 \supset A_1$, with the relevant branching rules $\Lambda_1 \downarrow (5/2)$, $\Lambda_2 \downarrow (2) + (4)$, and $2\Lambda_1 \downarrow (1) + (3) + (5)$. The corresponding R-matrix is identified with the $S = 5/2$ member of (3). Likewise, all the other $sp(2n)$-invariant R-matrices (9) can be identified with members of (3). This corrects a mis-identification in [1].

In case (iv) the R-matrix is

$$\begin{aligned}\check{R}^{\Lambda_2,\Lambda_2}(u) = {} & (1-6u)(1-u)(1-\tfrac{3}{2}u)P_{2\Lambda_2} + (1+6u)(1-u)(1-\tfrac{3}{2}u)P_{\Lambda_1} \\ & +(1+6u)(1+u)(1-\tfrac{3}{2}u)P_0 + (1-6u)(1-u)(1+\tfrac{3}{2}u)P_{\Lambda_2}.\end{aligned} \tag{10}$$

The maximal embedding is $G_2 \supset A_1$ and the relevant branching rules are $\Lambda_2 \downarrow (3)$, $2\Lambda_2 \downarrow (2) + (4) + (6)$ and $\Lambda_1 \downarrow (1) + (5)$. Therefore the G_2-invariant R-matrix (10) is identified with the extra $S = 3$ solution (6).

To summarise, we have the following identification of $su(2)$-invariant spin-S R-matrices with R-matrices invariant under a larger algebra and acting on tensor products of lowest

† References to the original sources in the cases where the R-matrices were discovered earlier in other forms can be found therein.

dimensional irreps:

$$\begin{aligned}
&\check{R}_{\rm I}^{2S\Lambda_1,2S\Lambda_1}\ [\,su(2)\,] = \check{R}^{\Lambda_1,\Lambda_1}\ [\,su(2S+1)\,] && (S \text{ half integer})\\
&\check{R}_{\rm IIa}^{2S\Lambda_1,2S\Lambda_1}\ [\,su(2)\,] = \check{R}^{\Lambda_1,\Lambda_1}\ [\,so(2S+1)\,] && (S \text{ integer})\\
&\check{R}_{\rm IIb}^{2S\Lambda_1,2S\Lambda_1}\ [\,su(2)\,] = \check{R}^{\Lambda_1,\Lambda_1}\ [\,sp(2S+1)\,] && (S \text{ half odd integer})\\
&\check{R}_{\rm V}^{6\Lambda_1,6\Lambda_1}\ [\,su(2)\,] = \check{R}^{\Lambda_2,\Lambda_2}\ [\,G_2\,] && (S=3).
\end{aligned}$$

The R-matrices $\check{R}_{\rm III}^{2S\Lambda_1,2S\Lambda_1}[su(2)]$ are already in the 'proper' Lie algebraic setting—corresponding to the trivial embedding of A_1 in itself. Finally, $\check{R}_{\rm IV}^{2S\Lambda_1,2S\Lambda_1}[su(2)]$ is not rational in u, unlike the others under consideration, and is in a class of its own—being related to the Temperley–Lieb algebra. We have thus 'accounted for' all the R-matrices in [1].

We conclude with a few remarks on whether there are $su(2)$-invariant R-matrices beyond those in [1]. Firstly, we note that it is possible to construct R-matrices by way of the Temperley–Lieb algebra starting from a pair $(\mathcal{G}, \Lambda)$ for which $V_\Lambda \otimes V_\Lambda$ is multiplicity-free and contains the trivial representation V_0 [10]. However, if such an R-matrix turns out to be $su(2)$-invariant—i.e. condition (a) is satisfied—then it will necessarily be equivalent to (5). Secondly, an inspection of the tables of [4] will convince the reader that condition (a) holds very rarely. In particular, the R-matrices associated with the fundamental representations of D_n, E_6, E_7 and F_4 are not $su(2)$-invariant. We believe that the list (i)–(iv) exhausts all such situations. Granted this, it can be argued that all $su(2)$-invariant R-matrices and hence integrable spin chains have already been identified in [1].

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